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Steady-State Solutions of Discrete-Velocity Boltzmann Systems in Restricted Flow Regions

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Mathematics Research Center
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Abstract: The existence of steady-state solutions is established for discrete-velocity Boltzmann systems in a restricted flow region. The principal result states that such solutions exist and are positive provided that the boundary scattering operator does not distort the associated kinetic equilibrium solutions too much. The steady-state solutions are represented as perturbations of kinetic equilibrium solutions.

I. INTRODUCTION

To develop a discrete-velocity Boltzmann model, the classical one-particle distribution function $p(t,x,v)$, with spatial coordinate x restricted to a three-dimensional flow region Ω with boundary Γ, and with unrestricted velocity coordinate v, is replaced by n one-particle velocity-type distribution functions $p_k(t,x)$, $1 \leqslant k \leqslant n$, for n discrete-velocity types v_k, $1 \leqslant k \leqslant n$. The classical integrodifferential equation for determining $p(t,x,v)$ with mixed initial-boundary conditions is then replaced by a first-order symmetric hyperbolic quasi-linear system for determining $p_k(t,x)$, $1 \leqslant k \leqslant n$, of the general form:

$$\frac{\partial}{\partial t} p_k(t,x) + v^k \cdot \nabla_x p_k(t,x) = B_k(p,p), \qquad t > 0, \qquad x \in \Omega, 1 \leqslant k \leqslant n,$$

$$B_k(p,p) = \sum_{j,\ell} \{ v_{j\ell} \, \Lambda^k_{j\ell} \, p_j p_\ell - v_{k\ell} \, \Lambda^j_{k\ell} p_k p_\ell \},$$

with the initial condition

$$p_k(0,x) = f_k(x), \qquad x \in \Omega, 1 \leqslant k \leqslant n,$$

and a boundary condition of the form

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$$|\nu^k \cdot n(x)| p_k(t,x) = \sum_j{}' \; \Pi_{jk}(x) |\nu^j \cdot n(x)| p_j(t,x), \qquad t \geqslant 0,\; x \in \Gamma,$$

for those values of k for which $\nu^k \cdot n(x) < 0$. ($\sum_j{}'$ denotes a summation over those values of j for which $\nu_j \cdot n(x) > 0$, where $n(x)$ is the unit exterior vector to Ω at x on the boundary Γ.) The numbers υ_{jk}, Λ_{jk}^{ℓ}, the vectors ν^k, and the functions $\Pi_{j,k}$, f_k, $1 \leqslant j,\, k,\, \ell \leqslant n$, are given data for the model.

In this report, we shall develop some results on the steady-state solutions of discrete-velocity Boltzmann models. These results are for application in a companion report on the time-dependent solutions of such models. However, the methods used in the two studies are so different that each is separately reported.

In a classical work on the spatially homogeneous Boltzmann equation, T. Carleman has a result on the global existence of time-dependent solutions of a general discrete-velocity, spatially homogeneous, one-dimensional model, (1; p. 100). Later, in his collected works on the kinetic theory of a gas (2; appendix), Carleman discussed a simple two-state, one-dimensional, spatially dependent unrestricted model, which is now referred to as Carleman's model. I.I. Kolodner proved the global existence of positive solutions of Carleman's model and discussed how his methods of proof could be used to treat more general forms of the collision operator (3). Using a different method, we have proved the global existence of positive solutions of one-dimensional, unrestricted, discrete-velocity models with special quadratic collision operators (4).

In the area of applications, J.E. Broadwell has used the method of discrete velocities to describe the structure of a shock wave in a gas in which the molecules move in only six directions and at constant speeds (5) and to study the problem of shear flow (6).

A discussion of the appropriate boundary conditions for the classical Boltzmann equation can be found in either the book, *Rarefied Gas Dynamics*, by M.N. Kogin (7) or the book, *Mathematical Methods in Kinetic Theory*, by C. Cercignani (8). The book by Cercignani contains some results (Ch. VI, Sec. 1 through 5) on the solution of the steady-state restricted flow problem for the linearization of the classical Boltzmann equation with respect to a Maxwellian steady-state solution. Section 5 of Ch. VI also contains a summary of the work by Y. Pao on the steady-state solutions of the one-dimensional linear and weakly nonlinear flow between infinite parallel plates (9). J. P. Guirard has published an extensive examination of the solutions of restricted flow problems for the linearization of the classical Boltzmann equation with respect to a Maxwellian steady-state solution (10).

II. PRELIMINARY NOTATION

We shall assume that the following data are given:

Bounded flow region $\Omega \subset R^3$ with boundary Γ and unit exterior vector $n(x)$ at $x \in \Gamma$. (2.1a)

Velocity types $\nu^k \in R^3 - \{0\}$, for all $1 \leqslant k \leqslant n$, (2.1b)

Collision rates $\upsilon_{jk} = \upsilon_{kj} \in R_+$, for all $1 \leqslant j, k \leqslant n$, (2.1c)

Collision scattering laws $\Lambda_{jk}^{\ell} \in R_+$, for all $1 \leqslant j, k, \ell \leqslant n$, (2.1d)

Boundary scattering laws $\Pi_{jk}(x) \in R_+$, for all $1 \leqslant j, k \leqslant n$, (2.1e)

and $x \in \Gamma$ for which $\nu^k \cdot n(x) < 0$ and $\nu^j \cdot n(x) > 0$.

For any set E and for $x = R$ or R^n, we shall use

$F(E,X)$ to denote the collection of mappings from E to X. (2.2a)

$C^r(E,X)$ to denote those mappings from E to X which are r-times continuously differentiable over E. (2.2b)

$C(E,X)$ to denote the collection of bounded and continuous mappings from E to X. (2.2c)

For a given set of velocity types $\{\nu^k,\ 1 \leqslant k \leqslant n\}$, we define the mapping

$$S\colon\ C^1(\Omega,R^n) \to C(\Omega,R^n) : f \to Sf,$$

$$(Sf)_k(x) = \left(\nu^k \cdot \nabla f_k\right), \qquad 1 \leqslant k \leqslant n, \qquad x \in \Omega. \tag{2.3}$$

For a given scattering rate υ and collision scattering law Λ, we shall define the (nonlinear) collision operator B and state some of its intrinsic properties:

$$B : C(\bar{\Omega},R^n) \to C(\bar{\Omega},R^n) : \xi \to B(\xi,\xi) \tag{2.4}$$

$$(B(\xi,\xi))_k = \sum_{\ell,m} B^k_{\ell m}\, \xi_\ell \xi_m,$$

$$B^k_{\ell m} \equiv \frac{1}{2}\, \upsilon_{\ell m}\left(\Lambda^k_{\ell m} + \Lambda^k_{m\ell} - \delta(k,\ell)\sum_{j=1}^{n} \Lambda^j_{km} - \delta(k,m)\sum_{j=1}^{n} \Lambda^j_{k\ell}\right) \qquad 1 \leqslant k,\ell,m < n$$

$$B^k_{\ell m} = B^k_{m\ell}; \qquad B^k_{\ell m} \geqslant 0,\ \ell \text{ and } m \neq k, \tag{2.4a,b}$$

$$B^k_{\ell m} \leqslant 0,\ \ell \text{ or } m = k, \qquad \sum_{\ell,m} B^k_{\ell m} = 1, \tag{2.4c,d}$$

$$\sum_{k} B^k_{\ell m} = 1, \qquad 1 \leqslant \ell, m \leqslant n. \tag{2.4e}$$

For a given set of velocity types $\{\nu^k : 1 \leqslant k \leqslant n\}$ and flow region Ω, we shall define the various emitting, impinging, and tangent parts of Γ.

Let $n(x)$ be the unit exterior vector to Ω at x on Γ. For $1 \leqslant k \leqslant n$,

$$\Gamma^e(k) = \{x \in \Gamma : \nu^k \cdot n(x) < 0\}. \tag{2.5a}$$

$$\Gamma^i(k) = \{x \in \Gamma : \nu^k \cdot n(x) > 0\}. \tag{2.5b}$$

$$\Gamma^t(k) = \{x \in \Gamma : \nu^k \cdot n(x) = 0\} \tag{2.5c}$$

$$\Omega^+(k) = \Omega \cup \Gamma^e(k) \cup \Gamma^i(k). \tag{2.5d}$$

We shall introduce some special notation for various collections of n-tuples of functions $f = (f_1, \ldots, f_n)$, of which each f_k has a distinct domain of definition.

$$C(\Omega^+,R^n) = \{f = (f_1, \ldots, f_n) : f_k \in C(\Omega^+(k),R),\ 1 \leqslant k \leqslant n\}. \tag{2.6a}$$

$$C(\Gamma^e,R^n)=\{f=(f_1,\ldots,f_n):f_k\in C(\Gamma^e(k),R),\ 1\leqslant k\leqslant n\}. \tag{2.6b}$$

$$C(\Gamma^i,R^n)=\{f=(f_1,\ldots,f_n):f_k\in C(\Gamma^i(k),R),\ 1\leqslant k\leqslant n\}. \tag{2.6c}$$

For a given boundary scattering law Π, we can construct the boundary scattering operator R. We shall assume that the boundary scattering law Π is such that R has certain useful properties.

$$R:C(\Gamma^i,R^n)\to C(\Gamma^e,R^n):f\to Rf \tag{2.7}$$

$$(Rf)_k(x)=\sum_j{}' \Pi_{jk}(x)|\nu^j\cdot n(x)|\,|\nu^k\cdot n(x)|^{-1} f_j(x),\qquad x\in\Gamma^e(k). \tag{2.7a}$$

$$\left(\sum_j{}' \text{ denotes a summation over those values of } j \text{ for which } \nu^j\cdot n(x)>0.\right). \tag{2.7b}$$

R is a positivity preserving (linear) operator. (2.7c)

If the boundary scattering law Π is such that

$$\sum_k \Pi_{jk}(x)=1,\ \text{then}\ \sum_k \nu^k\cdot n(x)\,f_k(x)=0.$$

For any $f\in C(\Omega^+,R^n)$, we shall now define the restrictions of f to the emitting and impinging parts of Γ.

Suppose that $f\in C(\Omega^+,R^n)$ so that $f_k\in C(\Omega^+(k),R)$, $1\leqslant k\leqslant n$. (2.8)

$$E:C(\Omega^+,R^n)\to C(\Gamma^e,R^n), \tag{2.8a}$$

$$(Ef)_k=f_k\big|_{\Gamma^e(k)}\quad\text{over } \Gamma^e(k),\ 1\leqslant k\leqslant n.$$

$$I:C(\Omega^+,R^n)\to C(\Gamma^i,R^n), \tag{2.8b}$$

$$(IF)_k=f_k\big|_{\Gamma^i(k)}\quad\text{over } \Gamma^i(k),\ 1\leqslant k\leqslant n.$$

Before completing this section, we want to introduce some more notation.

Suppose f and $g\in C(\Omega^+,R^n)$: (2.9a)

$f=g$ over Ω^+ is an abbreviation for $f_k=g_k$ over $\Omega^+(k)$, $1\leqslant k\leqslant n$.

Suppose f and $g\in C(\Gamma^e,R^n)$ or $C(\Gamma^i,R^n)$: (2.9b)

$f=g$ over Γ^e or Γ^i is an abbreviation for $f_k=g_k$ over $\Gamma^e(k)$ or $\Gamma^i(k)$, $1\leqslant k\leqslant n$.

III. REFORMULATION

Using the notation and the definitions of Sec. II, we shall express the problem of steady-state solutions in the form:

$$Sg = B(q,q) \text{ over } \Omega. \tag{3.1a}$$

$$Eq = RIq \text{ over } \Gamma^e. \tag{3.1b}$$

The construction of B shows that

$$B(c1, c1) = c^2 B(1,1) = 0, \qquad 1 = (1, \ldots, 1),\ c \in R. \tag{3.2}$$

Consequently, for any $c > 0$, $c1$ is a positive solution of 3.1a. If 1 satisfies 3.1b, then $\{c1; c > 0\}$ is a one-parameter family of steady-state solutions which can be used to order-bound the class of non-negative data for the time-dependent problem; i.e., to each f in $C(\Omega, R^n_+)$ there corresponds a $\underline{c}$ and a $\bar{c}$ such that $\underline{c}1 < f_k \leqslant \bar{c}1$. In general, 1 will not satisfy 3.1b.

The boundary scattering operator R is called kinetically active (passive) if

$$\kappa_R = RI1 - EI1 \neq 0 \ (\equiv 0) \qquad \text{over } \Gamma^e. \tag{3.3}$$

Assuming that R is kinetically active, we shall look for a steady-state solution of 3.1 in the form

$$q \equiv c1 + g, \tag{3.4}$$

so that g is to be determined as a solution of

$$Sg = 2\,c\,B(1,g) + B(g,g), \qquad \text{over } \Omega, \tag{3.5a}$$

$$Eg = RIg + \delta \qquad \text{over } \Gamma^e, \tag{3.5b}$$

$$\delta = c\,\kappa_R \in C(\Gamma^e, R^n),\ c > 0. \tag{3.5c}$$

For convenience of notation, we shall define the linear collision operator:

$$\begin{gathered} L : C(\Omega^+, R^n) \to C(\Omega^+, R^n) : f \to Lf = 2B(1,f), \\ (Lf)_k = 2\sum_{j=1}^{n} L_{jk} f_j, \quad L_{jk} = 2\sum_{\ell=1}^{n} B^{\ell}_{jk}, \quad 1 \leqslant j,k \leqslant n. \end{gathered} \tag{3.6}$$

At this stage it is worthwhile to exploit a certain positivity property (2; Lemma 1) which applies to both B and L. Therefore, for an arbitrary diagonal positive operator

$$\begin{gathered} D : C(\Omega^+, R^n) \to C(\Omega^+, R^n) : f \to Df \\ (Df)_k = d_k f_k, \quad d_k > 0, \quad 1 \leqslant k \leqslant n, \end{gathered} \tag{3.7}$$

we have transformed the original problem 3.1 into finding a solution g of

$$(S + D)g = cLg + Dg + B(g,g) \quad \text{over } \Omega, \tag{3.8a}$$

$$Eg = RIg + \delta \quad \text{over } \Gamma^e, \tag{3.8b}$$

for arbitrary δ in $C(\Gamma^e,R^n)$. If g is a solution of 3.8, if Γ is an active boundary, and if δ is chosen to be of the form 3.5c, then $q = c1 + g$ is the solution of 3.1.

The next step is to reformulate 3.8 in a form which treats the relationship between g and δ over Ω equally to that over Γ^e.

Setting aside the details until the next section, we shall let Jh denote the solution of

$$(S + D)g = h \quad \text{over } \Omega, \quad h \in C(\Omega^+,R^n), \tag{3.9a}$$

$$Eg = 0 \quad \text{over } \Gamma^e, \tag{3.9b}$$

and let $M\alpha$ denote the solution of

$$(S + D)g = 0 \quad \text{over } \Omega, \tag{3.10a}$$

$$Eg = \alpha \quad \text{over } \Gamma^e, \quad \alpha \in C(\Gamma^e,R^n). \tag{3.10b}$$

Then the solution g of

$$(S + D)g = h \quad \text{over } \Omega, \tag{3.11a}$$

$$Eg = \alpha \quad \text{over } \Gamma^e, \tag{3.11b}$$

for given $h \in C(\Omega^+,R^n)$ and $\alpha \in C(\Gamma^e,R^n)$ is given by the representation

$$g = Jh + M\alpha. \tag{3.12}$$

If $\alpha \in C(\Gamma^e,R^n)$ and $h \in C(\Omega^+,R^n)$, then $M\alpha$ and Jh are defined over Ω^+ in the sense of formula 2.9.

We next construct a map

$$\begin{aligned} G &: C(\Gamma^e,R^n) \times C(\Omega^+,R^n) \times C(\Gamma^e,R^n) \to C(\Omega^+,R^n) \times C(\Gamma^e,R^n), \\ &: (\delta,g,\alpha) \to (h,\beta) \end{aligned} \tag{3.13}$$

using the rule

$$h = g - J(cLg + D\,g + B(g,g)) - M\alpha - M\delta \quad \text{over } \Omega^+$$

$$\beta = \alpha - RIJ(cL + D)g - RIJB(g,g) - RIM\alpha - RIM\delta \quad \text{over } \Gamma^e.$$

The map G has three properties which are important for our purpose. The first is:

If $h = 0$, then (3.14)

$$Eg = \alpha + \delta \quad \text{over } \Gamma^e \tag{3.14a}$$

$$Ig = IJ(cL + D)g + IJB(g,g) + IM\alpha + IM\delta \quad \text{over } \Gamma^e. \tag{3.14b}$$

The second is:

$$\text{If } \beta = 0, \text{ then } \alpha = RIJ(cL + D)g + RIJB(g,g) + RIM\alpha + RIM\delta \quad \text{over } \Gamma^e. \tag{3.15}$$

The third is:

$$G(0,0,0) = (0,0). \tag{3.16}$$

A direct consequence of 3.14 and 3.15 is that the implicit relation

$$G(\delta,g,\alpha) = (0,0)$$

implies that

$$g = J(cLg + Dg + B(gg)) + M\alpha + M\delta \quad \text{over } \Omega^+, \tag{3.17a}$$

$$Eg = RIg + \delta \quad \text{over } \Gamma^e. \tag{3.17b}$$

If each term in the right side of 3.17a admits the application of $(S + D)$, then applying $(S + D)$ shows that

$$Sg = cLg + B(g,g) \quad \text{over } \Omega.$$

IV. STEADY-STATE SOLUTIONS

In the previous section, we have replaced the problem of finding a solution g of 3.8 with that of finding a solution (g,α) of

$$G(\delta,g,\alpha) = G(0,0,0) = (0,0).$$

We shall now fix $\delta \in C(\Gamma^e,R^n)$ as a parameter and treat $g \in C(\Omega^+,R^n)$ and $\alpha \in C(\Gamma^e,R^n)$ as independent variables. Letting

$$(h,\beta) = G(\delta,g,\alpha), \tag{4.1}$$

we shall define the map

$$F_\delta : C(\Omega^+,R^n) \times C(\Gamma^e,R^n) \to F(\Omega^+,R^n) \times F(\Gamma^e,R^n) : (g,\alpha) \to (g - k, \alpha - \gamma), \tag{4.2}$$

where (k,γ) is the solution of

$$\begin{aligned} k - J(cLk + Dk) &= h + M\gamma \quad && \text{over } \Omega^+ \\ \gamma - RIJ(cL + D)k &= \beta + RIMk \quad && \text{over } \Gamma^e. \end{aligned} \tag{4.3}$$

We shall then show that for each deformation parameter δ there is a closed, spherical neighborhood of $g = 0$ and $\alpha = 0$ of radius $r^* = r^*(\delta)$ over which E_δ is a contraction map (with contraction constant $\mu(\delta)$) and that F_δ displaces the center $(0,0)$ by less than $(1 - \mu)r^*$. This is sufficient to show that F_δ has a (unique) fixed point (g,α). It then follows from the construction of F_δ that $G(\delta,g,\alpha) = (0,0)$, so that g is a solution of 3.17. If $g \in C^1(\Omega,R^n)$ then g is a solution of 3.8. If δ has the form 3.5c, then $q = c1 + g$ is a steady-state solution of 3.1.

V. INVERSION OF $(S + D)$

In this section, we shall develop the solution g of (see 2.5, 2.7, and 2.8)

$$(S + D)g = h \quad \text{over } \Omega,\ h \in C(\Omega^+, R^n). \tag{5.1a}$$

$$Eg = \alpha \quad \text{over } \Gamma^e,\ \alpha \in C(\Gamma^e, R^n). \tag{5.1b}$$

To accomplish this purpose, we shall restrict the class of admissible flow regions Ω.

Assumption 1. There is a C^1-function ψ with domain $D(\psi)$ containing Ω such that Ω and Γ are determined by ψ:

$$\Omega = \{x \in D(\psi) : \psi(x) < 0\}, \tag{5.2a}$$

$$\Gamma = \{x \in D(\psi) : \psi(x) = 0\}, \tag{5.2b}$$

$$\{\psi = c_1\} \subset \{\psi = c_2\}, \quad c_1 < c_2, \tag{5.2c}$$

$$|(\nabla\psi)(x)| \neq 0, \quad x \in \Gamma,\ r \geqslant 1. \tag{5.2d}$$

The unit exterior vector n along Γ for a flow region Ω satisfying 5.2 is given in the form $n = |(\nabla\psi)|^{-1}$ $(\nabla\psi)$ over Γ.

The first problem is to find a representation for the solution g of

$$(S + D)g = 0 \quad \text{over } \Omega, \tag{5.3a}$$

$$Eg = \alpha \quad \text{over } \Gamma^e, \quad \alpha \in C(\Gamma^e, R^n). \tag{5.3b}$$

Corresponding to each velocity type ν^k, we assign to each x in Ω a unique $\bar{x}^k$ in $\Gamma^e(k)$ by choosing $\bar{x}^k(x)$ to be the first intersection with Γ^e of the directed line through x in the direction $(-\nu^k)$. To be more precise, for each k, $1 \leqslant k \leqslant n$, and each $x \in \bar{\Omega}$, let Φ^k be the natural parameterization of the characteristic line $C^k(x)$ through x associated with the direction ν^k:

$$C^k(x) = \{y \in R^3 : y = x + s\nu^k,\ s \in R^1\}, \tag{5.4a}$$

$$\Phi^k : R \times \bar{\Omega} \to R^3 \text{ is the solution of} \tag{5.4b}$$

$$\frac{d}{dt} x(t) = \nu^k, \quad x(0) = x,\ t \in R,\ x \in \bar{\Omega}.$$

Using the parametrization Φ^k of C^k, we define the first hitting time τ^k of C^k with $\Gamma^e(k)$ and the hitting place $\bar{x}^k$.

$$\tau^k : \bar{\Omega} \to R_+ : x \to \tau^k(x), \quad x \in \bar{\Omega}, \tag{5.5a}$$

$$\tau^k(x) = \min\{s \geqslant 0 : x - s\nu^k \in \Gamma^e\}, \quad 1 \leqslant k \leqslant n.$$

$$\bar{x}^k : \bar{\Omega} \to \Gamma^e(k) : x \to \bar{x}^k(x), \quad x \in \bar{\Omega}, \tag{5.5b}$$

$$\bar{x}^k(x) = \Phi^k(-\tau^k(x), x), \quad 1 \leqslant k \leqslant n.$$

The following properties are easily verified. For $1 \leqslant k \leqslant n$,

$$\tau^k \text{ and } \bar{x}^k \text{ are continuous over } \Omega \cup \Gamma. \tag{5.5a}$$

$$\tau^k(x) = 0, \quad x \in \Gamma^e(k) \cup \Gamma^t(k). \tag{5.5b}$$

$$x^k(x) = x, \quad x \in \Gamma^e(k) \cup \Gamma^t(k). \tag{5.5c}$$

We want to extend the application of the free-flow operator S by interpreting S as a directional derivative. So, we define the following:

$\widetilde{D}(k)$ = the collection of functions f in $C(\Omega^+, R)$ for which the directional derivative (5.7a)

$$(\widetilde{S}_k f)(x) = \lim_{t \to 0} \frac{1}{t}\{f(x + t\nu^k) - f(x)\}$$

exists at each $x \in \Omega$, $1 \leqslant k \leqslant n$, and is continuous over Ω.

$$\widetilde{D} = \{f \in C(\Omega^+, R^n) : f_k \in \widetilde{D}(k),\ 1 \leqslant k \leqslant n\}. \tag{5.7b}$$

$$\widetilde{S} : \widetilde{D} \to C(\Omega, R^n) : f \to \widetilde{S}f \tag{5.7c}$$

$$(\widetilde{S}f)_k(x) = (\widetilde{S}_k f_k)(x), \quad x \in \Omega,\ 1 \leqslant k \leqslant n.$$

The directional derivative $\widetilde{S}$ is an extension of S in the sense that

$$C^1(\Omega, R^n) \subset \widetilde{D}, \tag{5.8a}$$

$$\widetilde{S}f = Sf,\ f \in C^1(\Omega, R^n), \quad \text{over } \Omega. \tag{5.8b}$$

PROPOSITION 1. *If the determining function ψ for Ω (see 5.2) is a continuous function, then*

(a) $\tau^k \in \widetilde{D}(k)$

(b) $(\widetilde{S}_k \tau^k) = 1$ over Ω, $\quad 1 \leqslant k \leqslant n$

(c) $f(\bar{x}^k) \in \widetilde{D}(k)$, $f \in C^1(\Omega, R)$, $\quad 1 \leqslant k \leqslant n$

(d) $\widetilde{S}_k(f(\bar{x}^k)) = 0$ $\quad$ over Ω, $f \in C^1(\Omega, R)$, $\quad 1 \leqslant k \leqslant n$.

Since the proof of Proposition 1 is an elementary exercise, it has been omitted.

PROPOSITION 2. *If the determining function ψ for Ω is a C^1-function over its domain, then*

(a) τ^k is a C^1-function and $\bar{x}^k$ is a C^1-map over Ω.

(b) $S^k = \nu^k \cdot \nabla$ can be applied to τ^k and $f(\bar{x}^k)$, $f \in C^1(\Omega, R)$, $\quad 1 \leqslant k \leqslant n$.

Proof. For fixed x^0 in Ω, the properties 5.3a, b, and c imply the existence of a (smallest) s^0 for which $\psi(x^0 - s^0\nu^k) = 0$. The functions τ^k and $\bar{x}^k$ evaluated at x^0 are related to s^0:

(a) $\tau^k(x^0) = s^0$.

(b) $\bar{x}^k(x^0) = x^0 - s^0 \nu^k$.

Setting $F(x,s) = \psi(\Phi^k(s,x))$, we have (for sufficiently small s)

(a) $F(x^0,s^0) = 0$.

(b) $\dfrac{\partial F}{\partial s}(x,s) = -\nu^k \cdot (\nabla\psi\,(x - s\,\nu^k)),\ x \in \Omega$.

Since

$$|\nabla\psi| \neq 0 \text{ over } \Gamma,\ \frac{\partial F}{\partial s}(x^0,s^0) \neq 0.$$

Using the classical implicit function theorem and the (real) analyticity of Φ^k, we find that

(a) there exists (i) a neighborhood $N(x^0)$ of x^0 and (ii) a function $s: N(x^0) \to R$, $s(x^0) = s^0$, for which $F(x,s(x)) = \psi\,(s - s(x)\nu^k) = 0$, $x \in N(x^0)$, and

(b) s has the same order of differentiability as ψ.

Using again the properties of ψ (5.2) and the definition of τ^k(5.5), we know that $\tau^k(x) = s(x)$ for $x \in N(x^0)$. Consequently, τ^k has the same order of differentiability over Ω as ψ. Since $\bar{x}^k(x) = \Phi^k(-\tau^k(x),x)$, $\bar{x}^k$ also has the same order of differentiability over Ω as ψ.■

Using the results of Propositions 1 and 2, we define for each diagonal operator D (3.7) the linear map

$$M: F(\Gamma^e,R^n) \to F(\Omega,R^n): \alpha \to M\alpha, \tag{5.9}$$

$$(M\alpha)_k(x) = e^{-d_k\tau^k(x)}\alpha_k(\bar{x}^k(x)), \qquad x \in \Omega^+(k),\ 1 \leqslant k \leqslant n.$$

With the properties of τ and $\bar{x}$ stated in 5.6 and the definition of M, 5.9, there is no difficulty in showing that M maps a bounded continuous α over Γ^e into a bounded continuous $M\alpha$ over Ω^+;

$$M: C(\Gamma^e,R^n) \to C(\Omega^+,R^n). \tag{5.10}$$

PROPOSITION 3. *Suppose* $\alpha \in C(\Gamma^e,R^n)$. *Then*

(a) $M\alpha \in \widetilde{D}$.

(b) $(\widetilde{S} + D)(M\alpha)(x) = 0, \quad x \in \Omega$

(c) $EM\alpha = \alpha \quad$ over Γ^e.

PROPOSITION 4. *Suppose* $\alpha \in C(\Gamma^e,R^n) \cap C^1(\Gamma^e,R^n)$. *Then*

(a) $M\alpha \in C^1(\Omega,R^n)$

(b) $(\widetilde{S} + D)(M\alpha) = (S + D)(M\alpha) \quad$ over Ω.

Since the verification of the results stated in Propositions 3 and 4 follows directly from 5.9, 5.10 and the results of Propositions 1 and 2 as a routine exercise in the application of S_k and $\widetilde{S}_k$, they have been omitted.

The next problem is to find a representation for the solution g of

$$(S + D)g = h \qquad \text{over } \Omega, \; h \in C(\Omega^+, R^n) \tag{5.11a}$$

$$Eg = 0 \qquad \text{over } \Gamma^e. \tag{5.11b}$$

To accomplish this purpose, we may use the properties of τ^k and $\bar{x}^k$ to define the linear operator

$$J: C(\Omega, R^n) \to F(\Omega^+, R^n) : h \to Jh$$

$$(Jh)_k(x) = e^{-d_k \tau^k(x)} \int_0^{\tau^k(x)} e^{d_k s}\, h_k(\bar{x}^k(x) + s\nu^k)\, ds, \qquad x \in \Omega^+(k), \qquad 1 \leqslant k \leqslant n. \tag{5.12}$$

We want to stress that $(Jh)_k(x)$ is defined as a weighted line integral of $h_k(s)$ (the weighting factor is $\exp \; -d_k(\tau^k(x) - s)$ over the directed segment from $\bar{x}^k(x)$ to x in the direction ν^k. Using 5.6 and the definition of J, 5.12, we find that J maps a bounded continuous h over Ω into a bounded continuous Jh over Ω^+.

$$J: C(\Omega, R_n) \to C(\Omega^+, R^n). \tag{5.13}$$

PROPOSITION 5. *Suppose $h \in C(\Omega, R^n)$. Then*

(a) $Jk \in \widetilde{D}$

(b) $(\widetilde{S} + D)(Jk) = k$ *over* Ω

(c) $EJh = 0$ *over* Γ^e.

PROPOSITION 6. *Suppose $h \in C(\Omega^+, R^n) \cap C^1(\Omega, R^n)$. Then*

(a) $Jh \in C^1(\Omega, R^n)$

(b) $(\widetilde{S} + D)Jh = (S + D)Jh$ *over* Ω.

Proof. As previously emphasized, $(Jh)_k(x)$ is defined as the evaluation of a line integral. Consequently, in Proposition 5, a and b are simply restatements of the fact that the directional derivative of an indefinite line integral in the direction tangent to the line results in a number equal to the integrand evaluated at the point of differentiation on the line. The validity of Proposition 5c follows directly from the definition of Jh (5.12).

Since τ^k is a C^1-function and $\bar{x}^k$ is a C^1-map when ψ is a C^1-function, the upper limit and the integrand in the line integral which defined Jh are both C^1. Therefore Jh is a C^1-function. Both 7a and b follow directly from this. ∎

We shall now define norms for the collections $C(\Omega^+, R^n)$, $C(\Gamma^e, R^n)$, and $C(\Gamma^i, R^n)$. The classical uniform norm of a bounded function f defined over a set A is denoted by $|f|_A$.

$$N(f)_{\Omega^+} = \max\,(|f_k|_{\Omega^+(k)};\quad 1 \leqslant k \leqslant n). \tag{5.14a}$$

$$N(f)_{\Gamma^e} = \max\,(|f_k|_{\Gamma^e(k)};\quad 1 \leqslant k \leqslant n). \tag{5.14b}$$

$$N(f)_{\Gamma^i} = \max\,(|f_k|_{\Gamma^i(k)};\quad 1 \leqslant k \leqslant n). \tag{5.14c}$$

Since each collection $C(\Omega^+(k),R)$, $C(\Gamma^e(k),R)$, and $C(\Gamma^i,R)$, $1 \leqslant k \leqslant n$, with its uniform norm is a B-space, each collection $C(\Omega^+,R^n)$, $C(\Gamma^e,R^n)$, and $C(\Gamma^i,R^n)$ is a B-space; and so, any product combination of $C(\Omega^+,R^n)$, $C(\Gamma^e,R^n)$, and $C(\Gamma^i,R^n)$ with the appropriate sum norm is a B-space.

PROPOSITION 7. *Suppose* $\alpha \in C(\Gamma^e,R^n)$. *Then*

$$|M\alpha_k|_{\Omega^+(k)} \leqslant |\alpha_k|_{\Gamma^e(k)}, \quad 1 \leqslant k \leqslant n. \tag{5.15}$$

PROPOSITION 8. *Suppose* $h \in C(\varphi,R^n)$. *Then*

$$|(Jh)_k|_{\Omega^+(k)} \leqslant \frac{1}{d_k}\left(1 - e^{-d_k|\tau^k|_{\Omega^+(k)}}\right)|h_k|_{\Omega^+(k)}, \quad 1 \leqslant k \leqslant n. \tag{5.16}$$

Using Propositions 7 and 8 and the definitions in 5.14 results in estimates for M and J:

$$N(M\alpha)_{\Omega^+} \leqslant N(\alpha)_{\Gamma^e} \tag{5.17a}$$

$$N(Jh)_{\Omega^+} \leqslant c(D,\Omega)N(h)_{\Omega^+} \tag{5.17b}$$

$$c(D,\Omega) = \max\left\{\frac{1}{d_k}\left(1 - e^{-d_k|\tau^k|_{\Omega^+(k)}}\right);\quad 1 \leqslant k \leqslant n\right\}.$$

Since these inequalities are direct consequences of the definitions of M and J, a verification of Propositions 7 and 8 and 5.17 has been omitted.

The preceding analysis for S, M, and J can be easily modified for application to the dual problem of finding a solution g^* of

$$(-S + D)g = h \in C(\Omega,R^n). \tag{5.1*a}$$

$$Ig = \alpha^* \in C(\Gamma^i,R^n). \tag{5.1*b}$$

The most significant change is to define the first hitting time τ^{k*} of C^k with $\Gamma^i(k)$ and the associated hitting place $\bar{x}^{k*}$.

$$\tau^{k*} : \bar{\Omega} \to R_+ : x \to \tau^{k*}(x),\ x \in \bar{\Omega}, \tag{5.5*a}$$

$$\tau^{k*}(x) = \min\{\, s > 0 : x + sv^k \in \Gamma^i \},\quad 1 \leqslant k \leqslant n.$$

$$\bar{x}^{k*} : \bar{\Omega} \to \Gamma^i(k) : x \to \bar{x}^{k*}(x),\quad x \in \bar{\Omega}, \tag{5.5*b}$$

$$\bar{x}^{k*}(x) = \Phi^k(\tau^{k*}(x),x),\quad 1 \leqslant k \leqslant n.$$

Replacing τ with τ^* and $\bar{x}$ with $\bar{x}^*$, we can apply the preceding development to 5.1*. In particular, the auxiliary operators M^* and J^* can be defined as follows:

$$M^* : C(\Gamma^i, R^n) \to C(\Omega^+, R^n) : \alpha \to M^*\alpha, \tag{5.9*}$$

$$(M^*\alpha)_k(x) = e^{d_k \tau^{k^*}(x)} \alpha_k(x^{k^*}(x)), \quad x \in \Omega^+(k),\ 1 \leqslant k \leqslant n.$$

$$J^* : C(\Omega, R^n) \to C(\Omega^+, R^n) : h \to J^*h, \tag{5.12*}$$

$$(J^*h)_k(x) = e^{-d_k \tau^{k^*}(x)} \int_0^{\tau^{k^*}(x)} e^{+d_k s} h_k\left(\bar{x}^{k^*}(x) - s v^k\right) ds, \quad x \in \Omega^+(k), \quad 1 \leqslant k \leqslant n.$$

Since there are no further significent changes, those propositions for M^*, J^*, and $S^* = -S$ corresponding to those for M, J, and S can be stated without verification.

PROPOSITION 3*. *Suppose* $\alpha \in C(\Gamma^i, R^n)$. *Then*

(a) $M^*\alpha \in \widetilde{D}$

(b) $(\widetilde{S}^* + D)(M^*\alpha)\ 0$ *over*

(c) $IM^*\alpha = \alpha$ *over* $\Gamma^i \cup \Gamma^t$.

PROPOSITION 4*. *Suppose* $\alpha \in C(\Gamma^i, R^n) \cap C^1(\Gamma^i, R^n)$. *Then*

(a) $M\alpha \in C^1(\Omega, R^n)$.

(b) $(\widetilde{S}^* + D)(M\alpha) = (S^* + D)(M\alpha)$, *over* Ω.

PROPOSITION 5*. *Suppose* $h \in C(\Omega^+, R^n)$. *Then*

(a) $J^*h \in \widetilde{D}$.

(b) $(\widetilde{S}^* + D)(J^*h) = h$, *over* Ω.

(c) $IJ^*h = 0$, *over* Γ^i.

PROPOSITION 6*. *Suppose* $h \in C(\Omega^+, R^n) \cap C^1(\Omega, R^n)$. *Then*

(a) $J^*h \in C^1(\Omega, R^n)$.

(b) $(\widetilde{S}^* + D)(J^*h) = (S^* + D)(J^*h)$, *over* Ω.

There are also estimates for $M^*\alpha$ and J^*h similar to those for $M\alpha$ and Jh as stated in Propositions 7 and 8. We shall not state them; however, we will refer to them as Propositions 7* and 8* when we need them.

It is also of interest to see how the associated structures stated in Propositions 3 through 6 and 3* through 6* can be applied to the question of the uniqueness of the solutions g and g^* of the dual problems 5.1 and 5.1*.

Proposition 9 (9*). *Suppose the determining function ψ for Ω is a C^1-function. If $g(g^*) \in C^1(\Omega, R^n)$ is a solution of 5.1 (5.1*) for $h = 0$ and $\alpha = 0$, then $g(g^*) = 0$.*

Proof. The proof is made only for S and not S^* and, in fact, it is made for the weaker version of 5.1 when S is replaced with $\widetilde{S}$.

First, we choose an arbitrary f in $C_0(\Omega, R^n)$. From Proposition 5*a, we know that $J^*f \in \widetilde{D}$. Therefore, we consider the expressions

$$(J^*f)_k \, \widetilde{S}_k g_k + g_k \, \widetilde{S}_k^*(J^*f)_k, \qquad 1 \leqslant k \leqslant n.$$

Since $\widetilde{S}_k^*(J^*f)_k = f_k$ over Ω and since $\widetilde{S}g = h = 0$ over Ω, we integrate these expressions over Ω and apply the divergence theorem to obtain the expressions

$$\int_\Omega g_k f_k \, d\Omega = \int_\Gamma (J^*f)_k \, g_k \, \nu^k \cdot n(x) \, d\Gamma, \qquad 1 \leqslant k \leqslant n.$$

Decomposing the right side using $\Gamma = \Gamma^e(k) \cup \Gamma^i(k) \cup \Gamma^t(k)$ yields

$$\int_\Omega g_k f_k d\Omega = \int_{\Gamma^i(k)} (J^*f)_k \big|_{\Gamma^i(k)} \, g_k \big|_{\Gamma^i(k)} \, \nu^k \cdot n(x) \, d\Gamma$$

$$\int_{\Gamma^e(k)} (J^*f)_k \big|_{\Gamma^e(k)} \, g_k \big|_{\Gamma^e(k)} \, \nu^k \cdot n(x) \, d\Gamma, \qquad 1 \leqslant k \leqslant n.$$

The first term in the right side is equal to zero since f has compact support in Ω. The second is equal to zero since $Eg = \alpha = 0$. Therefore,

$$\int_\Omega g_k f_k \, d\Omega = 0, \qquad 1 \leqslant k \leqslant n, \qquad f \in C_0(\Omega, R^n).$$

This is sufficient to conclude that $g_k = 0$ over Ω, $1 \leqslant k \leqslant n$. ■

Before concluding this section we should like to point out that an analysis of the related problems

$$(S + D)g - \lambda g = h \qquad \text{over } \varphi, \quad \lambda \in R_+ \tag{5.18a}$$

$$Eg = RIg + \delta \qquad \text{over } \Gamma^e \tag{5.18b}$$

$$(S^* + D)g^* - \lambda g^* = h, \qquad \text{over } \Omega, \quad \lambda \in R_+ \tag{5.19a}$$

$$Ig = REg + \delta \qquad \text{over } \Gamma^i \tag{5.19b}$$

is necessary for determining that $S + D$ $(S^* + D)$ can be restricted to an appropriate domain $D_R(D_{R^*}) \subset \widetilde{D}$ associated with the boundary restrictions 5.18b or 5.19b, so that it is the infinitesimal generator of a C^0-semigroup over the closure of $D_R(D_{R^*})$. This semigroup is used in the approach to developing the existence for all $t \geqslant 0$ of positive solutions of the time-dependent problem associated with 3.1. An analysis of 5.18 and 5.19 will be given in a future report on time-dependent solutions.

UNCLASSIFIED

VI. PROPERTIES OF G

In this section, we shall develop those properties of G (3.11) which are used to solve the implicit relation $G(\delta,g,\alpha) = (0,0)$. The map G is defined so that if

$$(h,\beta) = G(\delta,g,\alpha), \tag{6.1}$$

then

$$h = g - J(cLg + Dg + B(g,g)) - M\alpha - M\delta \qquad \text{over } \Omega^+, \tag{6.1a}$$

$$\beta = \alpha - RIJ(cL + D)g - RIJ_B(g,g) - RIM\alpha - RIM\delta \qquad \text{over } \Gamma^e. \tag{6.1b}$$

PROPOSITION 10. *If (i) δ, $\alpha \in C(\Gamma^e,R^n)$; (ii) $g \in C(\Omega,R^n)$, then the functions h and β defined in 6.1 satisfy*

(a) $h \in C(\Omega^+,R^n)$

(b) $\beta \in C(\Gamma^e,R^n)$

(c) $g - h \in \widetilde{D}$.

Proof. There is no difficulty in showing that $k = Lg + Dg + B(g,g) \in C(\Omega,R^n)$ if $g \in C(\Omega,R^n)$. Since $M\alpha$ and $M\delta \in C(\Omega^+,R^n)$ if α and $\delta \in C(\Gamma^e,R^n)$ (5.10), and $Jh \in C(\Omega^+,R^n)$ if $k \in C(\Omega,R^n)$ (5.13), each term in the right side of 6.1a belongs to $C(\Omega^+,R^n)$. This verifies 6.1a.

Using the additional facts that (i) the impinging restriction $If \in C(\Gamma^i,R^n)$ if $f \in C(\Omega^+,R^n)$ (2.8a) and (ii) $R\alpha \in C(\Gamma^e,R^n)$ if $\alpha \in C(\Gamma^i,R^n)$ (2.7), and repeating the preceding argument will show that $\beta \in C(\Gamma^e,R^n)$. This verifies 6.1b. Finally, using the facts that (i) $M\alpha \in \widetilde{D}$ if $\alpha \in C(\Gamma^e,R^n)$ and (ii) $Jk \in \widetilde{D}$ if $k \in C(\Omega,R^n)$, a second repetition of the same argument will show that $(g-h) \in \widetilde{D}$.∎

For fixed $\delta \in C(\Gamma^e,R^n)$, we shall let G_δ be the restriction of G to the δ-plane.

$$G_\delta : C(\Omega^+,R^n) \times C(\Gamma^e,R^n) \to C(\Omega^+,R^n) \times C(\Gamma^e,R^n) \tag{6.2}$$

$$G_\delta(g,\alpha) = G(\delta,g,\alpha).$$

VII. AUXILIARY ESTIMATES

In this section, we develop some estimates for the operators $J(cL + D)$ and RIM which appear in the definition of G (3.13). These estimates will be used in the construction of the auxiliary map F_δ (4.2).

Before making these estimates, some further restrictions on the linear collision operator L and on the boundary scattering operator R must be introduced.

Referring to the definition of L, 3.6, we assume that the diagonal part of L (which is always non-positive) is strictly negative. This restriction could also be achieved by imposing a suitable restriction on the scattering law Λ.

$$L_{kk} < 0, \qquad 1 \leqslant k \leqslant n. \tag{7.1}$$

Referring to the definition of R and recalling that $RI1$ is bounded over Γ^e reveals that any increase in the magnitude of $|v^k \cdot n(x)|^{-1}$ must be compensated for by the factor

$$\sum_j{}' r_{jk}(x).$$

Therefore, it is assumed that the product of these two factors is bounded. For a given flow region Ω and boundary scattering operator R, we define

$$\theta_k = \max\left\{\sum_j{}' r_{jk}(x)\,|v^k \cdot n(x)|^{-1} : x \in \Gamma^e(k)\right\}. \tag{7.2a}$$

$$\theta(R,\Omega) = \max\{\theta_k : 1 \leqslant k \leqslant n\}. \tag{7.2b}$$

We are interested only in those flow regions Ω and boundary scattering operators R for which

$$\theta(R,\Omega) < \infty, \text{ so that} \tag{7.3a}$$

$$N(R\alpha)_{\Gamma^e} \leqslant \theta(R,\Omega)N(\alpha)_{\Gamma^i}, \quad \alpha \in C(\Gamma^i, R^n). \tag{7.3b}$$

We shall now isolate similar damping factors which appear in the definitions of M and J. For a given set of velocity states $\{v^k\}_1^n$ and any (positive) diagonal operator D, we define the following:

$$\chi_k = \max\left\{\sum_j{}' |v^j \cdot n(x)|\, e^{-d_j\tau^j(x)} : x \in \Gamma^e(k)\right\} \tag{7.4a}$$

$$\chi(D,\Omega) = \max\{\chi_k : 1 \leqslant k \leqslant n\} \tag{7.4b}$$

$$\zeta_k = \max\{1 - e^{-d_k\tau^k(x)} : x \in \Omega^+(k)\} \tag{7.4c}$$

$$\zeta(D,\Omega) = \max\{\zeta_k : 1 \leqslant k \leqslant n\}. \tag{7.4d}$$

We can easily develop the following bounds for χ and ζ:

$$0 < \zeta(D,\Omega) < 1 \tag{7.5a}$$

$$0 < \chi(D,\Omega) < |\#|_\Gamma\, d(\Omega) \tag{7.5b}$$

where

(i) $\#(x)$ is the number of impinging velocity types at $x \in \Gamma$

(ii) $d(\Omega)$ is the minimum diameter of Ω.

Using the bounds stated in 7.5 and imposing the restriction 7.1, we make the desired estimate for $J(cL + D)$, for an appropriately chosen D. Referring to the diagonal part of L, 7.1, we shall henceforth choose D to be

$$\begin{aligned} &D : C(\Omega^+, R^n) \to C(\Omega^+, R^n) : f \to Df \\ &(Df)_k = d_k f_k, \quad d_k = -cL_{kk}, \quad 1 \leqslant k \leqslant n. \end{aligned} \tag{7.6}$$

Having chosen D as defined in 7.6, we simplify the notation by setting $K = cL + D$;

$$K : C(\Omega^+, R^n) \to C(\Omega^+, R^n) : f \to Kf = (cL + D)f. \tag{7.7}$$

PROPOSITION 11. *Suppose L satisfies 7.1 and D is as defined in 7.6. Then for any g in* $C(\Omega^+, R^n)$

$$N(JKg)_{\Omega^+} < \zeta(D,\Omega)\, N(g)_{\Omega^+}. \tag{7.8}$$

Proof. We shall let $h = Kg$, so that

$$h_k = c \sum_{j \neq k} L_{jk} g_j, \qquad 1 \leqslant k \leqslant n.$$

Since

$$L1 = 0, \; d_k = c \sum_{j \neq k} L_{jk};$$

and so

$$|h_k|_{\Omega^+(k)} \leqslant d_k N(g)_{\Omega^+}, \qquad 1 \leqslant k \leqslant n. \tag{7.9}$$

Using the definition of M (5.9), we have

$$|(Jh)_k(x)| \leqslant \frac{1 - e^{-d_k \tau^k(x)}}{d_k} \, |h_k|_{\Omega^+(k)}, \; x \in \Gamma^e(k), \qquad 1 \leqslant k \leqslant n. \tag{7.10}$$

Inserting 7.9 into the right side of 7.10 and using the definition of ζ (7.4), we have

$$|(Jh)_k(x)| \leqslant \left(1 - e^{-d_k \tau^k(x)}\right) N(g)_{\Omega^+} \leqslant \zeta_k N(g)_{\Omega^+}, \qquad x \in \Gamma^e(k), \; 1 \leqslant k \leqslant n.$$

This is certainly sufficient to conclude that 7.8 is valid.■

Using the bounds stated in 7.5 and imposing the restriction 7.3, we shall make the desired estimate for RIM.

PROPOSITION 12. *Assume 7.3 is satisfied. Then for any* $\alpha \in C(\Gamma^e, R^n)$,

$$N(RIM\,\alpha)_{\Gamma^e} \leqslant \theta(R,\Omega)\, \chi(D,\Omega)\, N(\alpha)_{\Gamma^e}. \tag{7.11}$$

Proof. Using the definition of R (2.7) yields

$$|(RIM\,\alpha)_k(x)| \leqslant \sum_j{}' \, r_{jk}(x) |\nu^k \cdot n(x)|^{-1} \, |\nu^j \cdot n(x)| \, |(IM\,\alpha)_j(x)|$$

$$x \in \Gamma^e(k), \qquad 1 \leqslant k \leqslant n.$$

Therefore using 7.3 to estimate the right side, we have

$$|(RIM\,\alpha)_k(x)| \leqslant \theta_k \sum_j{}' \, |\nu^j \cdot n(x)| \, |(IM\,\alpha)_j(x)|, \tag{7.12}$$

$$x \in \Gamma^e(k), \; 1 \leqslant k \leqslant n.$$

Using the definition of M (5.9) and the definition of χ_k (7.4) results in

$$\sum_j{}' |\nu^j \cdot n(x)| |(IM\,\alpha)_j(x)| \leqslant \sum_j{}' |\nu^j \cdot n(x)|\, e^{-d_j\tau^j(x)} |\alpha_j(x^j(x))| \leqslant \chi_k N(\alpha)_{\Gamma^e}, \quad x \in \Gamma^e(k),\ 1 \leqslant k \leqslant n. \tag{7.13}$$

Using 7.13 to estimate the right side of 7.12, we have

$$|(RIM\,\alpha)_k(x)| \leqslant \theta_k\, \chi_k N(\alpha)_{\Gamma^e}, \quad x \in \Gamma^e(k),\ 1 \leqslant k \leqslant n.$$

This is sufficient to conclude that 7.11 is valid. ∎

If X is a B-space, we can denote the identity map on X by I_d:

$$I_d : X \to X : x \to x. \tag{7.14}$$

PROPOSITION 13. *If D is as defined in 7.6 and if L satisfies 7.1, then the equation*

$$h = k - JKk \quad \text{over } \Omega^+ \tag{7.15}$$

has a unique solution k $(= (I_d - JK)^{-1} h)$ in $C(\Omega^+, R^n)$ for each h in $C(\Omega^+, R^n)$. Moreover,

$$N(k)_{\Omega^+} \leqslant (1 - \zeta(D,\Omega))^{-1} N(h)_{\Omega^+}. \tag{7.16}$$

Since the verification of this result follows directly from the estimate 7.8 as a standard result in the theory of bounded contraction operators, it has been omitted.

PROPOSITION 14. *Suppose R and Ω are such that $\theta(R,\Omega)\chi(D,\Omega) < 1$. Then the equation*

$$\beta = \gamma - RIM_\gamma \quad \textit{over } \Gamma^e \tag{7.17}$$

has a unique solution γ $(= (I_d - RIM)^{-1}\beta)$ in $C(\Gamma^e, R^n)$ for each β in $C(\Gamma^e, R^n)$. Moreover,

$$N(\gamma)_{\Gamma^e} \leqslant (1 - \theta(R,\Omega)\chi(0,\Omega))^{-1} N(\beta)_{\Gamma^e}. \tag{7.18}$$

The verification of this result follows directly from the estimate 7.11 for the same reasons as stated in the comment following 7.16.

VIII. CONSTRUCTION OF F_δ

Using the estimates developed in the preceding section, we shall proceed with the construction of the auxiliary map F_δ (4.2), which we shall restate in a form more suitable for our purpose. We emphasize that D and K are henceforth as defined in 7.6 and 7.7.

We shall define the auxiliary linear map T:

$$\begin{aligned} T : C(\Omega^+, R^n) \times C(\Gamma^e, R^n) &\to F(\Omega^+, R^n) \times F(\Gamma^e, R^n) \\ : (k,\gamma) &\to (h,\beta) \end{aligned} \tag{8.1a}$$

$$h = k - JKk - M\gamma \qquad \text{over } \Omega^+$$
$$\beta = \gamma - RIM\gamma - RIJKk \qquad \text{over } \Gamma^e. \tag{8.1b}$$

The mapping properties of R, I, L, D, M, and J which are stated in 2.7, 2.8, 3.6, 3.7, 5.10, and 5.13 are sufficient to show that T maps a continuous pair (k,γ) into a continuous pair (h,β).

$$T: C(\Omega^+,R^n) \times C(\Gamma^e,R^n) \to C(\Omega^+,R^n) \times C(\Gamma^e,R^n). \tag{8.2}$$

To show that T has an inverse, we shall prepare some constructions for solving two special subsystems of 8.1b.

Proposition 15. *Suppose R and Ω are such that $\theta(R,\Omega)\chi(D,\Omega) < 1$. Then to each (h,β) in $C(\Omega^+,R^n) \times C(\Gamma^e,R^n)$ there corresponds a unique solution (k,γ) in $C(\Omega^+,R^n) \times C(\Gamma^e,R^n)$ of the system*

$$h = k - M\gamma \qquad \text{over } \Omega^+$$
$$\beta = (I_d - RIM)\gamma \qquad \text{over } \Gamma^e. \tag{8.3}$$

Moreover, k and γ are determined by

$$k = h + M(I_d - RIM)^{-1}\beta \qquad \text{over } \Omega^+.$$
$$\gamma = (I_d - RIM)^{-1}\beta \qquad \text{over } \Gamma^e. \tag{8.4}$$

Proof. The second equation in 8.3 can be solved for γ when β is given, using Proposition 14. So, γ is given by the second expression in 8.4. Substituting this expression for γ into the first equation of 8.3 determines that k is given by the first expression in 8.4.

Proposition 16. *Suppose that R and Ω are such that for some λ, $0 < \lambda < 1$,*

$$\theta(R,\Omega) < \lambda(1 - \zeta(D,\Omega)). \tag{8.5}$$

Then the equation

$$h = (I_d - JK)k - MRIk \qquad \text{over } \Omega^+ \tag{8.6}$$

has a unique solution $k(= (I_d - JK - MRI)^{-1}h)$ for each h in $C(\Omega^+,R^n)$. Moreover,

$$N(h)_{\Omega^+} \geqslant (1-\lambda)(1 - \zeta(D,\Omega))N(k)_{\Omega^+}. \tag{8.7}$$

Proof. Since 7.16 implies that

$$N((I_d - JK)_{\Omega^+} \geqslant (1 - \zeta(D,\Omega))N(k)_{\Omega^+} \tag{8.8}$$

and since 5.15, 7.3b, and 8.5 imply that

$$N(MRIk)_{\Omega^+} \leqslant \theta(R,\Omega)N(k)_{\Omega^+} \leqslant \lambda(1 - \zeta(D,\Omega))N(k)_{\Omega^+}, \tag{8.9}$$

we have

$$N((I_d - JK - MRI)k)_{\Omega^+} \geqslant N((I_d - JK)k)_{\Omega^+} - N(MRIk)_{\Omega^+}$$

$$\geqslant (1-\lambda)(1-\zeta(D,\Omega))N(k)_{\Omega^+}.$$

This establishes 8.7. It also implies that the equation of 8.6 has a unique solution k in $C(\Omega^+,R^n)$ for each h in the range of $(I_d - JK - MRI)$.

We shall now show that the range of $(I_d - JK - MRI)$ is the same as the range of $(I_d - JK)$. Since Proposition 13 implies that $(I - JK)$ is invertible over $C(\Omega^+,R^n)$, the range of $(I_d - JK)$ is all of $C(\Omega^+,R^n)$. If the range of $(I_d - JK - MRI)$ is not all of $C(\Omega^+,R^n)$, then for each d, $0 < d < 1$, there exists a g in $C(\Omega^+,R^n)$ such that

$$g \text{ is not in the range of } (I_d - JK - MRI). \tag{8.10a}$$

$$N(g)_{\Omega^+} = 1. \tag{8.10b}$$

$$N(g-f)_{\Omega^+} > d, \text{ for all } f \text{ in the range of } (I_d - JK - MRI). \tag{8.10c}$$

(This is a property first formulated by F. Riesz. A proof can be found in Ref. 11, p. 218.) However, setting $h = (I_d - JK)^{-1} g$ and using

$$N((I_d - JK)h - (I_d - JK - MRI)h)_{\Omega^+} \leqslant N(MRIh)_{\Omega^+} \leqslant \theta(R,\Omega)\,N(h)_{\Omega^+}$$

$$\leqslant \lambda(1-\zeta(D,\Omega))\,N(h)_{\Omega^+} \leqslant \lambda N(I_d - JK)\,h)_{\Omega^+}.$$

Therefore

$$N(g - (I_d - JK - MRI)h)_{\Omega^+} \leqslant \lambda N(g). \tag{8.11}$$

If we first choose d, $\lambda < d < 1$, and then use 8.10 to produce the associated g, we find that 8.10c and 8.11 contradict each other. Therefore, the range of $(I_d - JK - MRI)$ is the same as the range of $(I_d - JK)$, which is all of $C(\Omega^+,R^n)$. ∎

We could have easily made a proof for Proposition 16 using contraction theory; however, the present proof seems better since it shows that the range of $(I_d - JK - MRI)$ is the same as the range of $(I_d - JK)$, whatever the latter.

With these results, we are prepared to construct the inversion of T.

PROPOSITION 17. *Suppose that R and Ω are such that for some λ, $0 < \lambda < 1$, $\theta(R,\Omega) < \lambda(1-\zeta(D,\Omega))$ and $\theta(R,\Omega)\ \chi(D,\Omega) < 1$. Then the system 8.1b has a unique solution (k,γ) in $C(\Omega^+,R^n) \times C(\Gamma^e,R^n)$ associated with each (h,β) in $C(\Omega^+,R^n) \times C(\Gamma^e,R^n)$.*

Proof. Since $\theta(R,\Omega)\ \chi(D,\Omega) < 1$, we can apply Proposition 15 to solve the subsystem 8.3. We shall denote this solution by $(\bar{k},\bar{\gamma})$. Then, setting $\tilde{k} = k - \bar{k}$ and $\tilde{\gamma} = \gamma - \bar{\gamma}$ and subtracting 8.3 from 8.1b, we have reduced the original problem to that of finding a solution $(\tilde{k},\tilde{\gamma})$ of

$$\begin{aligned} JK\bar{k} &= (I_d - JK)\tilde{k} - M\tilde{\gamma} && \text{over } \Omega^+, \\ RIJK\bar{k} &= (I_d - RIM)\tilde{\gamma} - RIJK\tilde{k} && \text{over } \Gamma^e. \end{aligned} \tag{8.12}$$

Applying RJ to the first equation in 8.12 and then subtracting the second, we find that

$$\tilde{\gamma} = RI\tilde{k} \quad \text{over } \Gamma^e. \tag{8.13}$$

Therefore, we have further reduced the problem to that of finding a solution of the first equation in 8.12 when $\tilde{\gamma}$ is given by 8.13:

$$JK\bar{k} = (I_d - JK)\tilde{k} - MRI\tilde{k} \quad \text{over } \Omega^+. \tag{8.14}$$

Since $\theta < \lambda(1 - \zeta)$, we can apply the result of Proposition 16 to obtain the unique solution $\tilde{k}$ of 8.14 which is associated with $JK\bar{k}$.

Therefore,

$$k = \tilde{k} + \bar{k} \quad \text{over } \Omega^+,$$
$$\gamma = RI\tilde{k} + (I_d - RIM)^{-1}\beta \quad \text{over } \Gamma^e, \tag{8.15}$$

$$\bar{k} = h + m\,(I_d - RIM)^{-1}\beta \quad \text{over } \Omega^+,$$
$$\tilde{k} = (I_d - JK - MRI)^{-1} JK\bar{k} \quad \text{over } \Omega^+, \tag{8.16}$$

is the desired solution (k,γ) in $C(\Omega^+,R^n) \times C(\Gamma^e,R^n)$ which is associated with (h,β). ∎

Using Proposition 17, we shall define T^{-1}:

$$T^{-1} : C(\Omega^+,R^n) \times C(\Gamma^e,R^n) \to C(\Omega^+,R^n) \times C(\Gamma^e,R^n)$$
$$: (h,\beta) \to (k,\gamma), \tag{8.17}$$

where (k,γ) is determined by 8.15 and 8.16.

IX. PROPERTIES OF F_δ

Using G_δ (6.2) and T^{-1} (8.17), we shall simplify the expressions for the definition of F_δ (4.1, 4.2, and 4.3) and also state an elementary mapping property:

$$F_\delta : I_d - T^{-1} \circ G_\delta : C(\Omega^+,R^n) \times C(\Gamma^e,R^n) \to C(\Omega^+,R^n) \times C(\Gamma^e,R^n). \tag{9.1}$$

In this section, we shall develop those properties of F_δ which we will use to show the existence of a (unique) fixed pair (g,α) for F_δ when δ is sufficiently small.

The first property is an estimate on how far F_δ displaces the pair $(0,0)$.

PROPOSITION 18. *Let θ be as defined in 7.2, ζ as in 7.4, and λ as in 8.5. Set $(k,\gamma) = F_\delta(0,0)$. Then*

$$N(k)_{\Omega^+} = ((1-\lambda)(1-\zeta(D,\Omega)))^{-1}\, N(\delta)_{\Gamma^e}. \tag{9.2a}$$

$$N(\gamma)_{\Gamma^e} = ((1-\lambda)(1-\zeta(D,\Omega)))^{-1}\, \theta(R,\Omega)\, N(\delta)_{\Gamma^e}. \tag{9.2b}$$

Proof. Using the same reasoning which developed 8.13, we conclude that $\gamma = RIk$. Therefore, 9.2b is a direct consequence of the estimate 7.3a for R and the estimate 9.2a.

Setting $(g,\alpha) = (0,0)$ in the system 6.1 defining G_δ, we find that $h = -M\delta$ and $\beta = -RIM\delta$. Since $(k,RIk) = T^{-1}(h,\beta)$, (k,RIk) is determined by solving the equation

$$-M\delta = k - JKk - MRIk \quad \text{over } \Omega^+.$$

The solvability of this equation was developed in Proposition 16. Applying 8.7 establishes the validity of 9.2a. ■

Another relevant property of F_δ is the existence of a contraction constant over

$$B(r) = \left\{(g,\alpha) \text{ in } C(\Omega^+,R^n) \times C(\Gamma^e,R^n);\ N(g)_{\Omega^+} + N(\alpha)_{\Gamma^e} \leqslant r\right\}. \tag{9.3}$$

As a means to developing such a constant, we first develop an auxiliary estimate on the magnitude of $JB(g,f)$.

PROPOSITION 19. *Let c and ζ be as defined in 3.4 and 7.4. Then for every (g,k) in $C(\Omega^+,R^n) \times C(\Gamma^e,R^n)$*

$$N(JB(g,k))_{\Omega^+} \leqslant \frac{2\zeta(D,\Omega)}{c} N(g)_{\Omega^+} N(k)_{\Omega^+}. \tag{9.4}$$

Proof. Using the properties of B (2.4), we have

$$|(B(g,k))_j| \leqslant \sum_{\ell, m \neq j} B^j_{\ell m} |g_\ell|\,|k_m| + \sum_m |B^j_{jm}|\,|g_j|\,|k_m| + \sum_\ell |B^j_{\ell j}|\,|g_\ell|\,|k_j|. \tag{9.5}$$

For the first term $T(1)$ on the right side, we have

$$T(1) \leqslant \left\{\sum_{m \neq j} \left(\frac{1}{2} L_{jm} - B^j_{jm}\right)\right\} N(g)_{\Omega^+} N(k)_{\Omega^+}$$

since $2\sum_{\ell \neq j} B^j_{\ell m} = L_{jm}$ (3.6). Since $c\sum_{m \neq j} L_{jm} = d_j$ (7.6 and $L1 = 0$) and $-2c\sum_m B^j_{jm} = d_j$

$$T(1) \leqslant \frac{d_j}{c} N(g)_{\Omega^+} N(k)_{\Omega^+}. \tag{9.6}$$

For the second term $T(2)$, we have

$$T(2) \leqslant \left\{\left(-\sum_m B^j_{jm}\right)\right\} N(g)_{\Omega^+} N(k)_{\Omega^+},$$

and, since $-2c\sum_m B^j_{jm} = d_j$,

$$T(2) \leqslant \frac{d_j}{2c} N(g)_{\Omega^+} N(k)_{\Omega^+}. \tag{9.7}$$

Similarily for the third term $T(3)$, we have

$$T(3) \leqslant \frac{d_j}{2c} N(g)_{\Omega^+} N(k)_{\Omega^+}. \tag{9.8}$$

Substituting 9.6, 9.7 and 9.8 into 9.5 results in

$$|(B(g,k))_j| \leqslant \frac{2d_j}{c} N(g)_{\Omega^+} N(k)_{\Omega^+}. \tag{9.9}$$

Next, using the estimate 5.16 for J, the estimate 9.9, and the definition of ζ_j, yields

$$|(JB(g,k))_j| \leqslant \frac{1 - e^{-d_j |\tau^j|_{\Omega^+}}}{d_j} |(B(g,k))_j| \leqslant \frac{2\zeta_j}{c} N(g)_{\Omega^+} N(k)_{\Omega^+}.$$

The estimate 9.4 is then a direct consequence of the definition of ζ (7.4). ■

We want to represent $(F_\delta(\bar{g},\bar{\alpha}) - F_\delta(g,\alpha))$ in a form to which we can apply the estimates which we have already developed. We shall let

$$(k,\gamma) = F_\delta(\bar{g},\bar{\alpha}) - F_\delta(g,\alpha) \tag{9.10a}$$

$$(h,\beta) = T(k,\gamma). \tag{9.10b}$$

Using $I_d = T^{-1} \circ T$, we first express (k,γ) in the form

$$(k,\gamma) = T^{-1} \circ T(\bar{g}-g, \bar{\alpha}-\alpha) - T^{-1} \circ (G_\delta(\bar{g},\bar{\alpha}) - G_\delta(g,\alpha)),$$

so that

$$(h,\beta) = T(\bar{g}-g, \bar{a}-a) - (G_\delta(\bar{g},\bar{\alpha}) - G_\delta(g,a)).$$

Then writing out the systems defining T (8.1) and G_δ (6.1, 6.2), we find that

$$h = JB(\bar{g},\bar{g}) - JB(g,g) \qquad \text{over } \Omega^+. \tag{9.11a}$$

$$\beta = RIJB(\bar{g},\bar{g}) - RIJB(g,g) \qquad \text{over } \Gamma^e. \tag{9.11b}$$

Since

$$B(g,f-h) = B(g,f) - B(g,h),$$

we have

$$B(\bar{g},\bar{g}) - B(g,g) = B(\bar{g},\bar{g}-g) + B(\bar{g}-g,g).$$

Substituting this expression into 9.11, we have the desired expression for (h,β):

$$h = JB(\bar{g}-g,g) + JB(\bar{g},\bar{g}-g) \qquad \text{over } \Omega^+. \tag{9.12a}$$

$$\beta = RIJB(\bar{g}-g,g) + RIJB(\bar{g},\bar{g}-g) \qquad \text{over } \Gamma^e. \tag{9.12b}$$

We have now prepared sufficient information to develop a contraction constant for F_δ over $B(r)$.

PROPOSITION 20. *Let c be as defined in 3.4, θ as in 7.2, ζ as in 7.4, λ as in 8.5 and $B(r)$ as in 9.3. For any $(\bar{g},\bar{a})$ and (g,α) in $C(\Omega^+,R^n) \times C(\Gamma^e,R^n)$, let*

$$(k,\gamma) = F_\delta(\bar{g},\bar{a}) - F_\delta(g,\alpha).$$

$$(h,\beta) = T(k,\gamma).$$

Then

$$N(k)_{\Omega^+} \leq ((1-\lambda)(1-\zeta(D,\Omega)))^{-1} \frac{2\zeta(D,\Omega)}{c} \left(N(\bar{g})_{\Omega^+} + N(g)_{\Omega^+}\right) N(\bar{g}-g)_{\Omega^+}. \tag{9.13a}$$

$$N(\gamma)_{\Gamma^e} \leq \theta(R,\Omega)((1-\lambda)(1-\zeta(D,\Omega)))^{-1} \frac{2\zeta(D,\Omega)}{c} \left(N(g)_{\Omega^+} + N(g)_{\Omega^+}\right) N(\bar{g}-g)_{\Omega^+}. \tag{9.13b}$$

Proof. The estimate 9.13b will follow directly from the estimate 7.3 for R and the estimate 9.13a.

Since $\beta = RIh$ (9.12), we know that $\gamma = RIk$. Therefore h and k are related as in 8.6. Applying the estimate 8.7, we have

$$N(h)_{\Omega^+} \geq (1-\lambda)(1-\zeta(D,\Omega))\, N(k)_{\Omega^+}. \tag{9.14}$$

Using the expression 9.12a for h and the estimate 9.4 for JB, we have

$$N(h)_{\Omega^+} \leq \frac{2\zeta(D,\Omega)}{c} \left(N(\bar{g}-g)_{\Omega^+} N(g)_{\Omega^+} + N(\bar{g}-g)_{\Omega^+} N(\bar{g})_{\Omega^+}\right).$$

Substituting this estimate into 9.14 and multiplying by $((1-\lambda)(1-\zeta(D,\Omega)))^{-1}$, we have developed the estimate 9.13a.

With (k,γ) as defined in 9.10a, the estimates developed in Proposition 20 are sufficient for us to assert that for any $(\bar{g},\bar{\alpha})$ and (g,α) in $B(r)$:

$$\left(N(k)_{\Omega^+} + N(\gamma)_{\Gamma^e} \leq ((1-\lambda)(1-\zeta(D,\Omega))\right)^{-1} \frac{4\zeta(D,\Omega)r}{c} \left(N(\bar{g}-g)_{\Omega^+} + N(\bar{\alpha}-a)_{\Gamma^e}\right).$$

We shall let

$$a(\zeta,\lambda) = ((1-\lambda)(1-\zeta(D,\Omega)))^{-1}. \tag{9.15a}$$

$$b(c,\zeta,\lambda) = a(\zeta,\lambda)\,\frac{4\zeta(D,\Omega)}{c}\,. \tag{9.15b}$$

Then, collecting and summarizing the main achievement of this section, we have shown that

$$\text{if } k^0, \gamma^0 = F_\delta(0,0), \text{ then } \left(N(k^0)_{\Omega^+} + N(\gamma^0)_{\Gamma^e}\right) \leq 2\, a(\zeta,\lambda)\, N(\delta)_{\Gamma^e}. \tag{9.16a}$$

if (k,γ) is as defined in 9.10a, then for any $(\bar{g},\bar{\alpha})$ and (g,α) in $B(r)$ (9.3)

$$\left(N(k)_{\Omega^+} + N(\gamma)_{\Gamma^e}\right) \leq b(c,\zeta,\lambda)\, r\left(N(\bar{g}-g)_{\Omega^+} + N(\bar{\alpha}-\alpha)_{\Gamma^e}\right), \tag{9.16b}$$

where $a(\zeta,\lambda)$ and $b(c,\zeta,\lambda)$ are as defined in 9.15.

X. EXISTENCE OF STEADY-STATE SOLUTIONS

We shall now show that if κ_R (see 3.3)

$$\kappa_R = RI1 - E1 \text{ over } \Gamma^e. \tag{10.1}$$

is sufficiently small over Γ^e, then, subject to those restrictions on θ, χ, and ζ which were used to develop the estimates stated in 9.16, there exists a solution (g,α^θ) of $F_\delta(g,\alpha) = (g,\alpha)$.

PROPOSITION 21. *Let c be as defined in 3.4, θ as in 7.2, and χ and ζ as in 7.4. Suppose*

$$\theta(R,\Omega) < \lambda(1-\zeta(D,\Omega)) \text{ for some } \lambda,\ 0<\lambda<1 \tag{10.2a}$$

$$\theta(R\,|\,\Omega)\, \chi(D,\Omega) < 1. \tag{10.2b}$$

Let $a(\zeta,\lambda)$ and $b(c,\zeta,\lambda)$ be as defined in 9.15 and define the positive numbers r^0 and r^* as

$$r^0 = \frac{1}{8c\, a(\zeta,\lambda)\, b(c,\zeta,\lambda)}, \tag{10.3a}$$

$$r^* = \frac{1}{2b(c,\zeta,\lambda)}. \tag{10.3b}$$

Then to each κ_R with $N(\kappa_R)_{\Gamma^e} < r^0$ there corresponds a unique solution (g^*,α^*) in $B(r^*)$ of

$$F_\delta(g,a) = (g,\alpha),\ \delta = c\kappa_R.$$

Moreover,

$$N(g^*)_{\Omega^+} < \frac{c}{8}\left(\frac{1-\zeta(D,\Omega)}{\zeta(D,\Omega)}\right). \tag{10.4a}$$

$$(g^*,\alpha^*) \text{ is a continuous function of } \kappa_R \text{ for } N(\kappa_R)_{\Gamma^e} < r^0, \tag{10.4b}$$

reducing to $(g^*,\alpha^*) = (0,0)$ when $\kappa_R = 0$.

Proof. With $a(\zeta,\lambda)$ and $b(c,\zeta,\lambda)$ as defined in 9.15, we want to choose r and $N(\kappa_R)_{\Gamma^e}$ so that

$$\frac{2ca(\zeta,\lambda)\, N(\kappa_R)_{\Gamma^e}}{r} < 1 - b(c,\zeta,\lambda)\, r.$$

The parabola $y = r - br^2$ intersects the $\{y = 0\}$ – axis at $r = 0$ and $r = 1/b$ and has the maximum value of $\bar{y} = 1/46$ at $r = 1/2b$. With r^0 as defined in 10.4, we see that if $N(\kappa_R)_{\Gamma^e} < r^0$ then the line $y = 2ca(\zeta,\lambda)\, N(\kappa_R)_{\Gamma^e}$ intersects the parabola $y = r - br^2$ in the two points $0 < r_1 < r_2 < 1/b(c,\zeta,\lambda)$. Therefore

$$\frac{2ca(\zeta,\lambda)\, N(\kappa_R)_{\Gamma^e}}{r^*} < 1 - b(c,\zeta,\lambda)\, r^*$$

whenever $N(\kappa_R)_{\Gamma^e} < r^0$.

In the estimates stated in 9.16, replacing r with r^* as defined in 9.3 results in

if $(h^0, \gamma^0) = F_\delta(0,0)$ and if $N(\kappa_R)_{\Gamma^e} \leqslant r^0$, then (10.5a)

$$\left(N(k^0)_{\Omega^+} + N(\gamma^0)_{\Gamma^e}\right) < (1 - b(c,\zeta,\lambda)\, r^*)\, r^*$$

if (k, γ) is as defined in 9.10, then for any $(\bar{g}, \bar{\alpha})$ and (g, α) in $B(r^*)$ (10.5b)

$$\left(N(k)_{\Omega^+} + N(\gamma)_{\Gamma^e}\right) < (b(c,\zeta,\lambda)\, r^*)\, r^*.$$

The estimates given in 10.5 show that $F_{c\kappa_R}$ is a contraction mapping over $B(r^*)$ (with contraction constant equal to $b(c,\zeta,\lambda)\, r^*$) uniformly with respect to κ_R with $N(\kappa_R)_{\Gamma^e} < r^0$ and that $F_{c\kappa_R}$ displaces $(0,0)$ by a distance less than $(1 - b(c,\zeta,\lambda)\, r^*)\, r^*$.

Applying the uniform-contraction, fixed-point theorem, we can conclude that $F_{c\kappa_R}(g,\alpha) = (g,\alpha)$ has a unique solution (g^*, α^*) in $B(r^*)$.

Since F_δ is continuous in δ. the solution (g^*, α^*) is continuous in κ_R over $N(\kappa_R)_{\Gamma^e} < r^0$. The estimate 10.4a follows from estimating $1/2b\,(c,\zeta,\lambda)$ and 10.4b follows from Proposition 18.

Since T^{-1} exists, $(g^*, \alpha^*) = F_\delta(g^*, \alpha^*)$ implies that $G_\delta(g^*, \alpha^*) = (0,0)$. As already discussed in Section III (3.17), this implies that

$$g^* = JKg^* + JB(g^*, g^*) + M\alpha^* + M(c\kappa_R) \quad \text{over } \Omega^+. \tag{10.6a}$$

$$Eg^* = RIg^* + c\kappa_R \quad \text{over } \Gamma^e. \tag{10.6b}$$

Since each term of the right side of 10.6a is in $\tilde{D}$ (5.7), we can apply the operator $(\tilde{S} + D)$ (5.8). Therefore g^* is a (weak) solution of the system

(a) $\tilde{S}g = cLg + B(g,g)$ over Ω^+.

(b) $Eg = RIg + c\kappa_R$ over Γ^e.

Consequently,

$$q^* = c\,1 + g^* \tag{10.7}$$

is a (weak) steady-state solution of

$$\tilde{S}q = B(q,q) \qquad \text{over } \Omega^+. \tag{10.8a}$$

$$Eq = RIq \qquad \text{over } \Gamma^e. \tag{10.8b}$$

Since $N(g^*)_{\Omega^+} < c/6$ (10.4), we also know that

$$\frac{5c}{6}\,1 < q^* < \frac{7c}{6}\,1 \qquad \text{over } \Omega^+ \tag{10.9}$$

($f < h$ over Ω^+ is an abbreviation for $f_k(x) < h_k(x)$, $x \in \Omega^+(k)$, $1 \leq k \leq n$.)

Therefore formula 10.7, $c > 0$, generates a one-parameter family of (weak) positive steady-state solutions which can be used to order-bound initial datum:

To each f in $C(\Omega^+, R^n)$ there correspond two positive constants, $\underline{c} < \bar{c}$, and two associated steady-state solutions, $q = c1 + g$ and $q = c1 + g$, for which

$$\underline{q} \leq f \leq \bar{q} \qquad \text{over } \Omega^+.$$

XI. COMMENTS

In this report, we have developed the existence of a one-parameter family of (weak) positive steady-state solutions of 3.1. This family of solutions is represented in the form $q = c1 + g$, $c > 0$, where g is a solution of

(a) $(S + Dg) = Kg + B(g,g)$ over Ω^+.

(b) $Eg = RIg + c\kappa_R$ over Γ^e.

The family of solutions exists provided that κ_R is sufficiently small and 10.2 is valid.

We should like to suggest three problems. The first is to investigate whether or not solutions exist for all κ_R when 10.2 is valid.

The second problem is to determine whether g is uniformly different from zero throughout Ω or only near the boundary Γ. The estimate 10.4 for the magnitude of g suggests that q is not very different from $c1$, in the sense that $B(g,g) = Sg$ should have a small magnitude. In particular, we would like to have estimates on the magnitude of Sg over compact subsets of Ω and over closed neighborhoods of Γ.

The third problem is to determine whether or not there exists a correspondence between the solutions g of 11.1 and the solutions k of its linearization (at $g = 0$):

$$(S + D)k = Kk \qquad \text{over } \Omega^+. \tag{11.1a}$$

$$Ek = RIk + c\kappa_R \qquad \text{over } \Gamma^e. \tag{11.1b}$$

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13. ABSTRACT

The existence of steady-state solutions is established for discrete-velocity Boltzmann systems in a restricted flow region. The principal result states that such solutions exist and are positive provided that the boundary scattering operator does not distort the associated kinetic equilibrium solutions too much. The steady-state solutions are represented as perturbations of kinetic equilibrium solutions.

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